A model of electron propagation in Zwanziger's formulation of quantum electrodynamics

1989 J. Phys. A: Math. Gen. 22 4589

(http://iopscience.iop.org/0305-4470/22/21/021)

J. Phys. A: Math. Gen. 22 (1989) 4589-4598. Printed in the UK

A model of electron propagation in Zwanziger's formulation of quantum electrodynamics

Pierre Gaspard†

Faculté des Sciences, CP 231, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Brussels, Belgium

Received 26 April 1989

Abstract. We report a study of the infrared properties of the electron propagator in a framework which uses a gauge-invariant electromagnetic potential. The quantisation of the electromagnetic potential is performed in the Zwanziger formulation of quantum electrodynamics to avoid the usual infrared divergences. The Källen–Lehmann representation of the infrared electron propagator in the photon vacuum is shown to be well defined and convergent in this framework. Furthermore, we show that the probability density of an electron wavepacket propagating in the vacuum decreases like $(\text{time})^{-3}$, a well known property which is recovered here in quantum electrodynamics.

1. Introduction

The propagation of an electron in the vacuum is a difficult problem in quantum electrodynamics (QED) because of the coupling with the infrared virtual photons. As a consequence, the singularity of the electron propagator near the mass shell in momentum space is converted from a pole into a branch point when the electromagnetic coupling is switched on. This branch point singularity has appeared in several calculations of the infrared electron propagator since the early works in QED and it constitutes a fundamental difficulty. In particular, let us mention two problems.

(I) The propagator in momentum space calculated in the standard Gupta–Bleuler framework does not admit a well defined Källen–Lehmann representation (Zwanziger 1975), albeit that such a representation is important for the definition of the incoming and outgoing scattering states.

(II) In non-relativistic quantum mechanics, the amplitude of a wavepacket describing the propagation of a free stable particle decreases like the inverse $\frac{3}{2}$ power of the time. Accordingly, the probability density obeys the inverse-cube-law falloff property, which is shared with classical mechanics. However in QED, this fundamental property is not obeyed if the propagator has a branch point singularity near the mass shell. The pole singularity is the unique momentum space singularity which would permit the recovery of the inverse-cube-law falloff property in QED (Stapp 1983).

The purpose of the present paper is to show that it is possible to solve both problems in the light of recent works on the construction of charged states in QED (Morchio and Strocchi 1983, D'Emilio and Mintchev 1983, 1984). In these works, the charged

† Present address: The James Franck Institute, The University of Chicago, 5640 Ellis Avenue, Chicago, IL 60637, USA.

states are obtained as generalised coherent states which obey the Gauss law. This fundamental condition was not satisfied in the standard Gupta–Bleuler framework, which is probably at the origin of the two aforementioned problems. In order to be able to calculate non-perturbatively the electron propagator, we shall use the Zwanziger formulation of QED. That author showed that the introduction of Hertz potentials avoids the infrared divergences, which makes this formulation particularly useful (Zwanziger 1979).

Our general assumptions for the construction of the charged states are presented in § 2 with some reference to the Zwanziger formulation of QED. Our model of electron propagation is introduced in § 3. The electron propagator in the photon vacuum is calculated in § 4. The propagator in momentum space and the Källen–Lehmann representation are derived in § 5. The wavepacket propagation is discussed in § 6.

2. General assumptions

With the purpose of calculating non-perturbatively the electron propagator, we need the equation of motion for the quantised fermion field coupled with the quantised electromagnetic field

$$(\mathrm{i}\gamma \cdot \partial - m)\Psi = q\gamma^{\mu}\mathscr{A}_{\mu}\Psi. \tag{2.1}$$

To avoid the difficulties of the standard Gupta–Bleuler framework that we mentioned in the introduction, the electromagnetic potential is chosen as the following distribution-valued operator:

$$\mathscr{A}_{\mu}(x) = A_{\mu}(x) - \int \mathrm{d}^4 y \frac{\partial}{\partial x^{\mu}} J^{\nu}(y) A_{\nu}(x-y). \tag{2.2}$$

$A_{\mu}(x)$ is the electromagnetic potential operator of the standard Gupta–Bleuler framework. $J^{\nu}(y)$ is a real distribution which obeys

$$\partial \cdot J(y) = \delta^4(y). \tag{2.3}$$

This condition implies that $\mathscr{A}_{\mu}(x)$ and $\Psi(x)$ are invariant under a gauge transformation like

$$A_{\mu}(x) \to A_{\mu}(x) - \partial_{\mu}\Lambda(x) \tag{2.4}$$

where $\Lambda(x)$ is a real function vanishing at infinity. Consequently, the charged states constructed in this framework obey the Gauss law (cf Morchio and Strocchi 1983, D'Emilio and Mintchev 1984).

One possible solution of (2.3) which will fulfil our needs in the present paper is

$$J_{\mu}(y) = \frac{1+c}{2}\int_0^{+\infty} u_{\mu}\delta^4(y-u\tau)\,\mathrm{d}\tau - \frac{1-c}{2}\int_{-\infty}^{0} u_{\mu}\delta^4(y-u\tau)\,\mathrm{d}\tau. \tag{2.5}$$

This real distribution depends on five real parameters: the 4-vector u_{μ} and the scalar c. Consequently, the charged field operator Ψ, solution of (2.1), will also depend on these parameters. An inertial frame of reference is thus privileged once u_{μ} is introduced in the construction of a quantum charged state as discussed by Fröhlich *et al* (1979a, b).

In the following section, we are concerned with the behaviour of the fields at large distances. Asymptotically, the electromagnetic potential is the sum of an operator-like part which is a quantised free potential plus the classical Coulomb potential produced by the mean electron current. In the present paper, we treat the case of a single electron. Accordingly, we assume that the electron field does not interact with its own Coulomb potential, an assumption which has already been made by several authors (Kulish and Faddeev 1970, Zwanziger 1975, Papanicolaou 1976). The electromagnetic potential appearing in (2.1) is thus the quantised free electromagnetic potential.

To avoid the infrared divergences, the Gupta-Bleuler electromagnetic potential $A_\mu(x)$ is quantised in the Zwanziger formulation of QED where the inner product defining the photon state space has the form

$$\langle A(\varphi_1)\Omega, A(\varphi_2)\Omega\rangle$$
$$= \int d^4x\, d^4y\, \varphi_1^{\mu}(x)\varphi_2^{\nu}(y)\langle\Omega, A_\mu(x)A_\nu(y)\Omega\rangle$$
$$= \frac{1}{2}\int d^2\hat{k}\int_0^\infty d\omega\, \ln(a_Z\omega)\frac{\partial}{\partial\omega}(\omega^2\tilde{\varphi}_1^{\mu *}(k)\tilde{\varphi}_{2\mu}(k))_{k_0=|k|=\omega}$$
$$+\int\frac{d^3\xi_1}{\xi_1^0}\frac{d^3\xi_2}{\xi_2^0}\rho_1(\xi_1)K(\xi_1,\xi_2)\rho_2(\xi_2). \tag{2.6}$$

The parameter a_Z appears in the Wightman function of the Hertz potentials from which the vector potential A_μ is derived. Such a parameter breaks the scale invariance of the electromagnetic system as shown by Zwanziger (1979). $\rho(\xi)$ is the zero-frequency limit of the wavefunction $\tilde{\varphi}_\mu(k)$ defined by

$$\lim_{\omega\to 0}(\omega\tilde{\varphi}_\mu(k))_{k_0=\omega} = \frac{i}{(2\pi)^{3/2}}\int\frac{\xi_\mu}{\xi\cdot\hat{k}}\rho(\xi)\frac{d^3\xi}{\xi_0}. \tag{2.7}$$

We use the notation $k_\mu = \omega\hat{k}_\mu$ with $\hat{k}_0 = 1$ and $d^2\hat{k}$ for the infinitesimal element of spherical angle. The function K is defined by

$$K(\xi_1,\xi_2) = K_i(\psi) + K_f(\xi_1,\xi_2) \tag{2.8}$$

with

$$\xi_1\cdot\xi_2 = \cosh\psi \qquad \psi \geq 0 \quad \xi_1^2 = \xi_2^2 = 1 \tag{2.9}$$

$$K_i(\psi) = \frac{1}{(2\pi)^2}\frac{1}{\tanh\psi}\int_0^\psi\frac{s}{\tanh s}\,ds \tag{2.10}$$

$$K_f(\xi_1,\xi_2) = -\frac{1}{2(2\pi)^3}\int d^2\hat{k}\,\ln(\xi_1\cdot\hat{k})\frac{\xi_1\cdot\xi_2}{(\xi_1\cdot\hat{k})(\xi_2\cdot\hat{k})}. \tag{2.11}$$

On the class of wavefunctions $\{\varphi^\mu(k)\}$ which are regular at $\omega = 0$, the first term in (2.6) is equal to the standard inner product and the second term vanishes. The Zwanziger inner product remains finite on the class of wavefunctions $\{\varphi^\mu(k)\}$ with a $1/\omega$ singularity at $\omega = 0$. Apart from this choice of inner product, we define the photon Fock space as in the standard Gupta-Bleuler framework.

3. The propagation model

The purpose of our model is to describe the asymptotic propagation of one electron coupled with the quantised electromagnetic field. We assume that no other charge is present. We define the asymptotic Green operator of Feynman type which rules the propagation of the asymptotic charged states by

$$\mathcal{G}^{\mathrm{as}}_{ab}(x,y)=\sum_{s=1,2}\int\frac{\mathrm{d}^3p}{2E}[\theta(x_0-y_0)\mathcal{U}_a(x;\boldsymbol{p},s)\bar{\mathcal{U}}_b(y;\boldsymbol{p},s)$$
$$-\theta(y_0-x_0)\bar{\mathcal{V}}_b(y;\boldsymbol{p},s)\mathcal{V}_a(x;\boldsymbol{p},s)]. \tag{3.1}$$

The operator acts on the direct product of the one-fermion state space with the photon state space. $\mathcal{U}$ and $\mathcal{V}$ are spinors of the one-fermion state space but operators over the photon state space. The indices a and b label the components of the 4-spinors. In $\bar{\mathcal{U}}$ and $\bar{\mathcal{V}}$, the adjoint operation was carried out on the spinor and the operator parts of $\mathcal{U}$ and $\mathcal{V}$. The positive-energy solution is propagated from past to future and the reverse for the negative-energy solution with a minus sign taking account of the fermion character of the electron.

$\mathcal{U}$ and $\mathcal{V}$ describe an incoming electron or positron of given 4-momentum p and spin s. They are the asymptotic solutions of the Dirac equation (2.1) where $\mathcal{A}_\mu(x)$ is now the incoming quantised electromagnetic potential defined by (2.2) with A_μ replaced by A^{in}_μ. Assuming that the momentum of the electron does not change very much in the propagation, these asymptotic solutions can be calculated with the eikonal approximation. Furthermore, we assume that the spacetime point x is close to the classical trajectory $x=p\tau$. Consequently, the spinors $\mathcal{U}$ and $\mathcal{V}$ have the forms

$$\mathcal{U}(x;\boldsymbol{p},s)=\frac{\mathrm{e}^{-\mathrm{i}p\cdot x}}{(2\pi)^{3/2}}u(\boldsymbol{p},s)\otimes W(p\cdot x/m^2) \tag{3.2}$$

$$\mathcal{V}(x;\boldsymbol{p},s)=\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(2\pi)^{3/2}}v(\boldsymbol{p},s)\otimes W(p\cdot x/m^2). \tag{3.3}$$

The operator W is a spacetime-dependent operator acting on the photon Fock space and it is given by

$$W(p\cdot x/m^2)=T\exp\left(-\mathrm{i}q\int_{-\infty}^{p\cdot x/m^2}p^\mu\mathcal{A}^{\mathrm{in}}_\mu(p\tau)\,\mathrm{d}\tau\right). \tag{3.4}$$

$u(\boldsymbol{p},s)$ and $v(\boldsymbol{p},s)$ are the positive- and negative-energy solutions of the free Dirac equation in standard spinor notation with the normalisation of Zwanziger (1975).

For an outgoing configuration, a similar formula holds with $\mathcal{A}^{\mathrm{in}}$ replaced by $\mathcal{A}^{\mathrm{out}}$ and $-\infty$ by ∞.

The time-ordered product can be eliminated and (3.4) becomes

$$W(p\cdot x/m^2)=\exp(-\mathrm{i}\theta_\infty)\times U(p\cdot x/m^2) \tag{3.5}$$

where

$$U(p\cdot x/m^2)=\exp\left(-\mathrm{i}q\int\mathrm{d}^4y\,\varphi(y;x)\cdot A^{\mathrm{in}}(y)\right). \tag{3.6}$$

The factor $\exp(-\mathrm{i}\theta_\infty)$ is an infinite phase due to the integration of the retarded photon propagator. This phase is omnipresent but may be considered as a constant independent of x. Deriving unitary operator U from (3.4), we introduced the coherent factor $\varphi^\mu(y;x)$. Its Fourier transform is given by

$$\tilde{\varphi}_\mu(k;x)=\frac{1}{(2\pi)^{3/2}}\int\mathrm{e}^{\mathrm{i}k\cdot y}\varphi_\mu(y;x)\,\mathrm{d}^4y$$
$$=\frac{-\mathrm{i}}{(2\pi)^{3/2}}\exp\left(\frac{\mathrm{i}k\cdot pp\cdot x}{m^2}\right)\left(\frac{p_\mu}{k\cdot p-\mathrm{i}0}-\frac{1-c}{2}\frac{u_\mu}{k\cdot u+\mathrm{i}0}-\frac{1+c}{2}\frac{u_\mu}{k\cdot u-\mathrm{i}0}\right) \tag{3.7}$$

where we used the definition (2.5) of $J_\mu(x)$.

The normal form of the unitary operator U is

$$U=\exp\left(\frac{-q^2}{2}N(x,x)\right)\exp\left(-\mathrm{i}q\int \mathrm{d}^4y\,\varphi(y;x)\cdot A^{(+)}(y)\right)$$

$$\times\exp\left(-\mathrm{i}q\int \mathrm{d}^4y\,\varphi(y;x)\cdot A^{(-)}(y)\right) \tag{3.8}$$

where the free electromagnetic potential A^{in} was separated into its positive- and negative-frequency parts, $A^{(+)}$ and $A^{(-)}$. $N(x,x)$ is defined as the limit $x\to y$ of the commutator

$$N(x,y)=\int \mathrm{d}^4z\,\mathrm{d}^4z'[\varphi(z;x)\cdot A^{(-)}(z),\varphi(z';y)\cdot A^{(+)}(z')]$$

$$=\langle A^{\text{in}}[\varphi(\;;x)]\Omega, A^{\text{in}}[\varphi(\;;y)]\Omega\rangle \tag{3.9}$$

where the spaces in $\varphi(\;;x)$ mean that each corresponding variable of φ is integrated in $A^{\text{in}}[\varphi]$ contrary to the other variable of φ. The expression (3.9) is calculated with the Zwanziger inner product (2.6). The integral over ω in the first term of (2.6) is made convergent in the ultraviolet by replacing $\mathrm{i}p\cdot(x-y)$ by $\varepsilon+\mathrm{i}p\cdot(x-y)$ (Zwanziger 1975). The result is

$$N(x,y)=\frac{2}{(2\pi)^2}\ln\left[\frac{\varepsilon+\mathrm{i}p\cdot(x-y)}{ma_Z\,\mathrm{e}^{-\gamma}}\right](1-\psi\coth\psi)$$

$$+\frac{1}{(2\pi)^2}\left(1+\psi\coth\psi-2\coth\psi\int_0^\psi s\coth s\,\mathrm{d}s\right) \tag{3.10}$$

where ψ is defined by

$$\cosh\psi=\frac{u\cdot p}{\sqrt{u^2p^2}}\qquad \psi\geqslant 0 \tag{3.11}$$

and γ is the Euler constant. We note that $N(x,y)$ is independent of the parameter c.

The unitary operator U maps the photon Fock space on another state space orthogonal to the Fock space. Indeed, if Φ_{ph} and Φ'_{ph} are two photon Fock states,

$$\langle\Phi'_{\text{ph}}, U\Phi_{\text{ph}}\rangle=0 \tag{3.12}$$

in the limit $\varepsilon\to 0$, because we have the inequality

$$\psi\coth\psi-1\geqslant 0. \tag{3.13}$$

This property is well known in the formulation using the standard inner product (Kulish and Faddeev 1970). The preceding calculation shows that it also holds in the Zwanziger formulation where the infrared divergences are regularised. The normal form (3.8) and the commutator (3.9) are used in the following section.

4. Electron propagation in the vacuum

The Green operator (3.1) describes an electron or a positron propagating in spacetime either remaining in the photon vacuum or with emission or absorption of light. The amplitudes of all these processes are contained in (3.1). Hereafter, we shall focus on the amplitude for the charged particle to propagate in the vacuum from the origin to x, given by

$$G^{as}(x)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E}\langle\Omega,\ W(p\cdot x/m^2)W^{+}(0)\Omega\rangle(\gamma\cdot p+m)\,e^{-ip\cdot x}\qquad x_0>0 \tag{4.1}$$

where Ω is the vacuum of the photon Fock space. From the expression (3.4) for W and using the normal form (3.8) for U, we obtain

$$\begin{aligned}\langle\Omega,\ &W(p\cdot x/m^2)W^{+}(0)\Omega\rangle\\ &=\exp\left(-\frac{q^2}{2}N(x,x)\right)\exp\left(-\frac{q^2}{2}N(0,0)\right)\exp(q^2N(x,0))\\ &=\left(\frac{\varepsilon}{\varepsilon+ip\cdot x}\right)^{\beta(p,u)}\end{aligned} \tag{4.2}$$

with the definition

$$\beta(p,u)=\beta(\psi)=\frac{2\alpha}{\pi}(\psi\coth\psi-1) \tag{4.3}$$

where ψ is the hyperbolic angle between u and p defined by (3.11). As expected from the unitarity of W, the expression (4.2) tends to 1 and x goes to zero. Similar expressions hold if $x_0<0$.

The Green function can then be written as

$$G^{as}(x)=\frac{i}{(2\pi)^4}\int d^4p\frac{\gamma\cdot p+m}{p^2-m^2+i0}\,e^{-ip\cdot x}\left(\frac{\varepsilon}{\varepsilon+ip\cdot x}\right)^{\beta(p,u)}. \tag{4.4}$$

This expression is independent of the parameter c, but depends on the 4-vector u_μ.

5. The propagator in momentum space

Using the integral representation

$$\frac{1}{(\varepsilon+iA)^\beta}=\frac{1}{\Gamma(\beta)}\int_0^\infty r^{\beta-1}\exp[-(\varepsilon+iA)r]\,dr \tag{5.1}$$

we obtain the asymptotic Green function in momentum space

$$\begin{aligned}G^{as}(p)&=\int d^4x\,e^{ip\cdot x}G^{as}(x)\\ &=i\int_0^\infty dr\frac{e^{-\varepsilon r}}{(1+r)^3}\frac{\varepsilon^{\beta'}}{\Gamma(\beta')}r^{\beta'-1}\frac{\gamma\cdot p+m(1+r)}{p^2-m^2(1+r)^2+i0}\end{aligned} \tag{5.2}$$

where

$$\beta'=\beta\left(\frac{p}{1+r},u\right)=\beta(\psi)=\beta(p,u). \tag{5.3}$$

This equality arises because β depends only on the hyperbolic angle between the two 4-vectors p and u and not on their length (see (3.11) and (4.3)). This remarkable property enables us to integrate easily. By the change of variable $M = m(1+r)$, the Källen–Lehmann representation of the electron propagator is finally obtained

$$G^{\mathrm{as}}(p) = \frac{\mathrm{i}\varepsilon^{\beta(p,u)}}{\Gamma[\beta(p,u)]} m^2 \int_m^{\infty} \frac{\mathrm{d}M}{M^3} \left(\frac{m}{M-m}\right)^{1-\beta(p,u)} \times \exp\left[-\varepsilon\left(\frac{M-m}{m}\right)\right] \frac{\gamma\cdot p + M}{p^2 - M^2 + \mathrm{i}0}. \tag{5.4}$$

Several comments are now in order. In the standard Gupta–Bleuler formalism with $J_\mu = 0$, the non-perturbative calculation of the Green function (Zwanziger 1975) led to a similar expression but with an exponent $-\alpha/\pi$ instead of β. With such a negative exponent, the integral over M diverges at $M = m$ and the Källen–Lehmann representation is not defined. Thanks to the positive sign of β (see (3.13)), the integral (5.4) converges and the representation exists. The positivity of β in the present framework has its origin in the transversality

$$k^\mu \tilde{\varphi}_\mu = 0 \tag{5.5}$$

of the coherent factor (3.7). The property (5.5) also implies the Gauss law, which holds in the present framework based on the gauge-invariant electromagnetic potential (2.2) (Morchio and Strocchi 1983, D'Emilio and Mintchev 1983, 1984). We may conclude here that the charged states must satisfy the Gauss law if we want the propagator to have a well defined Källen–Lehmann representation.

The integral over M in (5.4) is performed as follows. In the limit where ε goes to zero, the exponential disappears from the integral. This latter can then be transformed into a contour integral in the complex plane of the variable M. The singularity on the mass shell is thus

$$G^{\mathrm{as}}(p) \simeq \Gamma(1-\beta) \frac{\mathrm{i}(\gamma\cdot p + m)}{p^2 - m^2 + \mathrm{i}0} \left(\frac{\varepsilon}{2m^2}(m^2 - p^2 - \mathrm{i}0)\right)^{\beta(p,u)} \qquad p^2 \simeq m^2. \tag{5.6}$$

We now observe that $\beta(p, u)$ goes to zero when the momentum p goes to u, and the usual free propagator is recovered

$$G^{\mathrm{as}}(p) \simeq \frac{\mathrm{i}(\gamma\cdot p + m)}{p^2 - m^2 + \mathrm{i}0} \qquad p^2 \simeq m^2. \tag{5.7}$$

The infinitesimal parameter ε is also eliminated in this limit so that the transition amplitude does not vanish when $u = p$.

However, the physical meaning of the limit $u = p$ is not clearly apparent at this level. In the following section, we shall provide the full justification of a similar limit in the propagation where the initial and the final electron states are completely specified.

6. Wavepacket propagation

Here we consider the propagation of the electron wavepacket in the photon vacuum. We assume that the positron processes are negligible.

Our aim is to calculate the transition amplitude between states $\psi_1(x)$ and $\psi_2(x - X)$ where $X = (t, t\boldsymbol{v})$ is a 4-vector. The 3-vector $\boldsymbol{v}$ is fixed but t is a variable parameter taking values between 0 and ∞. $\psi_1(x)$ and $\psi_2(x)$ are 4-spinors localised near the spacetime origin $x = 0$. $\psi_1(x)$ describes a preparing device. $\psi_2(x - X)$ represents a detector movable in space along the axis $\boldsymbol{v}$ and detecting events delayed by a tunable time t. Such an experiment will select electrons with a mean velocity $\boldsymbol{v}$. Because the electron is a stable particle the transition amplitude is expected to decrease like $t^{-3/2}$ when the detector is separated from the preparing device (Stapp 1983). We now turn to the proof of this property.

The transition amplitude of the described process is defined by

$$J_{1\to2}(X) = \int d^3x_1\, d^3x_2\, \psi_2^\dagger(x_2 - X) G^{as}(x_2 - x_1)\gamma_0\psi_1(x_1) \tag{6.1}$$

using the Green function (4.4). We first integrate over p_0 in (4.4). Using the integral representation (5.1) and after integration over $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, the amplitude becomes

$$J_{1\to2}(X) = \int_0^\infty dr \int \frac{d^3p}{2E} \frac{\varepsilon^\beta}{\Gamma(\beta)} r^{\beta-1} e^{-\varepsilon r} \times \exp[-ip\cdot X(1+r)][\tilde{\psi}_2^\dagger(p(1+r))(\gamma\cdot p+m)\gamma_0\tilde{\psi}_1(p(1+r))] \tag{6.2}$$

where $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are the Fourier transforms of the corresponding 4-spinors.

We are interested in the asymptotic behaviour of this amplitude as the time t goes to ∞. We use the method of stationary phase to integrate over the 3-momentum $\boldsymbol{p}$ in this limit (Bleistein and Handelsman 1986). t appears in an oscillatory kernel $\exp(-it\phi)$ with the phase $\phi(\boldsymbol{p}) = (1+r)[(m^2+\boldsymbol{p}^2)^{1/2} - \boldsymbol{v}\cdot\boldsymbol{p}]$. The stationary point is $\boldsymbol{p} = m\boldsymbol{v}/\sqrt{1-\boldsymbol{v}^2} \equiv \boldsymbol{P}$. It is convenient to introduce the corresponding energy $P_0 \equiv m/\sqrt{1-\boldsymbol{v}^2}$.

Thereafter, the integral over r is performed with the same method. The stationary point is now $r = 0$.

After both integrations, we obtain the asymptotic behaviour of the transition amplitude for $t\to\infty$

$$J_{1\to2}(X) \simeq \left(\frac{2\pi P_0}{it}\right)^{3/2} \left(\frac{\varepsilon}{\varepsilon + im^2t/P_0}\right)^{\beta(P,u)} \times \exp(-im^2t/P_0)\left(\tilde{\psi}_2^\dagger(P)\frac{\gamma\cdot P+m}{2m}\gamma_0\tilde{\psi}_1(P)\right) \tag{6.3}$$

whereupon the transition probability of the described process decreases with time t like

$$\frac{\varepsilon^{2\beta(P,u)}}{t^{3+2\beta(P,u)}}. \tag{6.4}$$

The 4-momentum P is fixed by the relative position of the measuring devices 1 and 2. The 4-vector u was introduced in the construction of the charged quantum state in (2.5). But it remained arbitrary for the Källen–Lehmann representation (5.4) to be well defined. ε is the ultraviolet cutoff of the calculation. It is an infinitesimal quantity so that the transition probability (6.4) would be vanishing if $\beta(P, u)$ were not zero. This observation leads us to conclude that we have to choose $u = P$, because $\beta(P, u)$ is zero in this limit (see (3.11) and (4.3)). Under this condition, the fundamental inverse-cube-law falloff property is recovered in quantum electrodynamics.

Let us recall that the transition probability (6.4) concerns the propagation of the electron in the photon vacuum. When $u \neq P$, the propagation from device 1 to detector 2 is evanescent in the photon vacuum because of ε. But had we assumed that the electron is accompanied by a quasiclassical electromagnetic radiation field, we would have obtained a non-vanishing probability for some $u \neq P$. In order to obtain the transition probability of this more complicated process, we should consider the matrix elements of the Green operator (3.1) between photon coherent states, rather than between photon Fock states as in (4.1).

7. Conclusions

In the present paper, we calculated the infrared electron Green function in a simple model of electron propagation. We used the Zwanziger formulation of QED, which avoids the infrared divergences. It is then possible to recover two fundamental properties of the electron propagation, with the help of the gauge-invariant electromagnetic potential defined by (2.2) and (2.3).

(I) The Källen–Lehmann representation of the infrared Green function exists. This result holds because the charged states satisfy the Gauss law in the present framework.

(II) The inverse-cube-law falloff of the probability density is recovered under specific conditions.

The invariant electromagnetic potential we defined in § 2 depends on a 4-vector u_μ and on a scalar c. The property (I) holds for arbitrary values of these parameters. The parameter c remains arbitrary throughout the discussion of §§ 4–6 because it does not appear in the physical quantities. On the other hand, the 4-vector u_μ does appear in physical observables such as transition probabilities. In the discussion of property (II), we had to specify the initial and the final electron and photon states. At this stage we showed that the 4-vector u_μ may not remain arbitrary but has to be fixed by the configuration of the measuring devices in the experiment we want to describe. This important result shows that the charged states strongly depend on the measurement process in QED. One of the purposes of the present paper was to clarify this relationship in some simple example. Already in the early works on infrared problems in QED, the cross sections were shown to depend on a parameter ΔE characterising the non-zero energy resolution of the detector. The present formalism is now able to deal consistently with electron wavefunction as shown in § 6. It thus allows a characterisation of the measurement device by more detailed quantities than only a simple parameter like ΔE.

This conclusion is obtained in the present paper for a simplified model of electron propagation. Nevertheless, we believe that this conclusion would remain valid in other models of electron propagation sophisticated enough to incorporate the radiative corrections, because our conclusion is founded on the general assumptions of § 1. Similar results hold for the propagation of several charged particles as we shall show in a forthcoming publication.

Acknowledgments

It is my pleasure to thank Professor J Naisse who is at the origin of the present work. I thank him for his helpful advice. The author is 'Chargé de Recherches' at the National Fund for Scientific Research (Belgium).

References

Bleistein N and Handelsman R A 1986 *Asymptotic Expansions of Integrals* (New York: Dover) ch 6
D'Emilio E and Mintchev M 1983 *Phys. Rev.* D **27** 1840

D'Emilio E and Mintchev M 1984 *Fortschr. Phys.* **32** 473
Fröhlich J, Morchio G and Strocchi F 1979a *Ann. Phys., NY* **119** 241
—— 1979b *Phys. Lett.* **89B** 61
Kulish P and Faddeev L 1970 *Theor. Math. Phys.* **4** 745
Morchio G and Strocchi F 1983 *Nucl. Phys.* B **211** 471
Papanicolaou N 1976 *Phys. Rep.* **24** 229
Stapp H P 1983 *Phys. Rev.* D **28** 1386
Zwanziger D 1975 *Phys. Rev.* D **11** 3481
—— 1979 *Phys. Rev.* D **19** 3614